1. Let $\{X_n; n \geq 0\}$ be the simple symmetric random walk on $\mathbb{Z}$ starting at 0. For $x \in \mathbb{Z}$, let $\tau_x := \inf\{n : X_n = x\}$. For $a < 0 < b$, show that

$$P\{\tau_a < \tau_b\} = \frac{b}{b-a}.$$

Solution: For $a < 0 < b$,

$$\tau_a := \inf\{n : X_n = a\}, \quad \tau_a := \inf\{n : X_n = a\}.$$

Now, $\forall a < 0$,

$$\begin{aligned} P\{\tau_a < \tau_b\} &= P\{\tau_{a+1} < \tau_b\} \cdot P\{\tau_{-1} < \tau_{b-a-1}\} \\ &= P\{\tau_{a+2} < \tau_b\} \cdot P\{\tau_{-1} < \tau_{b-a-2}\} \cdot P\{\tau_{-1} < \tau_{b-a-1}\} \\ &= P\{\tau_{-1} < \tau_b\} \cdots P\{\tau_{-1} < \tau_{b-a-1}\}. \end{aligned} \tag{1}$$

Claim: $P\{\tau_{-1} < \tau_b\} = \frac{b}{b+1}$. We will prove it by induction. Now,

$$\begin{aligned} P\{\tau_{-1} < \tau_b\} &= 1 - P\{\tau_{-1} > \tau_b\} \\ &= 1 - P\{\tau_{-1} > \tau_{b-1}\} \cdot P\{\tau_1 < \tau_{-b}\} \\ &= 1 - P\{\tau_{-1} > \tau_{b-1}\} \cdot P\{\tau_{-1} < \tau_b\} \text{ [symmetric random walk, } P\{\tau_1 < \tau_{-b}\} = P\{\tau_{-1} < \tau_b\}] \end{aligned}$$

$$\begin{aligned} \Rightarrow P\{\tau_{-1} < \tau_b\} &= \frac{1}{1 + P\{\tau_{-1} > \tau_{b-1}\}} \\ &= \frac{1}{1 + 1 - P\{\tau_{-1} < \tau_{b-1}\}} \\ &= \frac{1}{1 + 1 - \frac{b-1}{b}} \text{ [by induction, } P\{\tau_{-1} < \tau_{b-1}\} = \frac{b-1}{b}] \\ &= \frac{b}{b+1}. \end{aligned}$$

Therefore, from (1)

$$\begin{aligned} P\{\tau_a < \tau_b\} &= P\{\tau_{-1} < \tau_b\} \cdots P\{\tau_{-1} < \tau_{b-a-1}\} \\ &= \frac{b}{b+1} \frac{b+1}{b+2} \cdots \frac{b-a-1}{b-a} \\ &= \frac{b}{b-a}. \end{aligned}$$

Alternative Solution:- Let $\tau_a = \inf\{t \geq 0 : X_t = a\}$, $\tau_b = \inf\{t \geq 0 : X_t = b\}$ and $\tau_{a,b} = \tau_a \wedge \tau_b$. Now, $\tau_{a,b}$ is a stopping time. Let $A = \{\tau_{a,b} = \tau_a\}$ be the event where X hits a before hitting b. We will compute $P(A)$. Since, $\{X_n\}$ is a simple symmetric random walk on $\mathbb{Z}$, hence $\limsup_{n\to\infty} X_n =$

∞ and $\liminf_{n\to\infty} X_n = -\infty$. Therefore, almost surely $\tau_a < \infty$ and $\tau_b < \infty$. By the Optional Stopping theorem, $X^{\tau_{a,b}}$ is a martingale. Since $\tau_{a,b} \wedge n \to \tau_{a,b}$ as $n \to \infty$ a.s., we get $X_n^{\tau_{a,b}} \to X^{\tau_{a,b}}$ a.s.. As $|X_n^{\tau_{a,b}}|$ is bounded by $b-a$, therefore $X_n^{\tau_{a,b}} \to X^{\tau_{a,b}}$ also in L^1. Thus

$$\begin{aligned}
0 &= \lim_{n\to\infty} \mathbb{E}[X_n^{\tau_{a,b}}] = \mathbb{E}[X^{\tau_{a,b}}] \\
&= a \cdot P[\tau_{a,b} = \tau_a] + b \cdot P[\tau_{a,b} = \tau_b] \\
&= a \cdot P[\tau_{a,b} = \tau_a] + b \cdot (1 - P[\tau_{a,b} = \tau_a]) \\
&= b + (a-b) P[\tau_{a,b} = \tau_a].
\end{aligned}$$

Therefore, $P[\tau_{a,b} = \tau_a] = \frac{b}{b-a}$. Hence we can conclude that $P\{\tau_a < \tau_b\} = \frac{b}{b-a}$.

□

2. Let $\{X_n; n \geq 0\}$ be a square integrable martingale in its natural filtration, with square variation process $\{< X >_n; n \geq 0\}$. Let τ be a finite stopping time (in the natural filtration of X) such that $\mathbb{E} < X >_\tau < \infty$. Show that

$$\mathbb{E}(X_\tau - X_0)^2 = \mathbb{E} < X >_\tau \quad \text{and} \quad \mathbb{E}X_\tau = \mathbb{E}X_0.$$

Solution: Given that $\{X_n; n \geq 0\}$ be a square integrable martingale, therefore $X_n^2 - < X >_n$ is also a martingale. Let us denote,

$$Z_n := X_n^2 - < X >_n .$$

Now, τ be a finite stopping time and Z_n is a martingale, therefore $Z_n^\tau := Z_{n\wedge\tau}$ is also a martingale. Without loss of generality, let us assume that $X_0 = 0$. Therefore,

$$\begin{aligned}
&\mathbb{E}[Z_{n\wedge\tau}] = 0 \\
&\Rightarrow \mathbb{E}[X_{n\wedge\tau}^2] = \mathbb{E}[< X >_{n\wedge\tau}].
\end{aligned} \tag{2}$$

Now, as $n \to \infty$, $< X >_{n\wedge\tau} \to < X >_\tau$. And by assumption, $\mathbb{E} < X >_\tau < \infty$. Therefor by Lebesgue DCT, $\mathbb{E}[< X >_{n\wedge\tau}] \to \mathbb{E}[< X >_\tau]$.
Let

$$\mathcal{F}_\infty := \sigma(\cup_{n\in\mathbb{N}_0} \mathcal{F}_n).$$

Since, $\{X_n; n \geq 0\}$ is a square integrable martingale with square variation process $\{< X >_n; n \geq 0\}$. Hence, $\sup_n \mathbb{E}[X_n^2] < \infty$. Again, by martingale convergence theorem for L^2, there exists an $\mathcal{F}_\infty$-measurable r.v. X_∞ with $\mathbb{E}[|X_\infty|^2] < \infty$ and $X_n \to X_\infty$ a.s. and in L^2. And $(|X_n|^2)_{n\in\mathbb{N}_0}$ is uniformly integrable. Therefore by Lebesgue DCT, we can conclude that, as $n \to \infty$, $\mathbb{E}[X_{n\wedge\tau}^2] \to \mathbb{E}[X_\tau^2]$.
Therefore, from (2), we can conclude that $\mathbb{E}[X_\tau^2] = \mathbb{E}[< X >_\tau]$, hence $\mathbb{E}[(X_\tau - X_0)^2] = \mathbb{E}[< X >_\tau]$, since we assumed w.l.o.g. $X_0 = 0$.

For the proof of the 2nd part, we proceed as follows- Since, τ is a finite stopping time (in the natural filtration of X) such that $\mathbb{E} < X >_\tau < \infty$. Therefore by applying Optional Stopping Theorem, we can conclude that $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

□

3. Let $f : [0, 1) \to \mathbb{R}$ be an integrable function w.r.t. Lebesgue measure λ. Let $I_{n,k} := [k2^{-n}, (k+1)2^{-n})$ for $n \in \mathbb{N}$ and $k = 0, 1, \cdots, 2^n - 1$. Define $f_n : [0, 1) \to \mathbb{R}$ as follows: if $x \in I_{n,k}$

$$f_n(x) := 2^n \int_{I_{n,k}} f d\lambda.$$

Show that $\{f_n; n \geq 1\}$ is a uniformly integrable martingale in an appropriate filtration and deduce that $f_n(x) \to f(x)$ for λ almost all $x \in [0, 1)$.

Solution: Consider,

$$\mathcal{F}_n := \sigma(\{I_{n,k} : k = 0, 1, \cdots, 2^n - 1)\}.$$

We will show that $\mathcal{F}_n$ is a filtration i.e. $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, where $\mathcal{F}_{n+1} := \sigma(\{I_{n+1,k} : k = 0, 1, \cdots, 2^{n+1} - 1)$. Now,

$$\begin{aligned}
I_{n,k} &:= [k2^{-n}, (k+1)2^{-n}) \\
&= [2k2^{-(n+1)}, (2k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)}) \\
&= I_{n+1,2k} \cup I_{n+1,2k+1}.
\end{aligned}$$

As $I_{n+1,2k}, I_{n+1,2k+1} \in \mathcal{F}_{n+1}$, therefore $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$.
f is a random variable on $([0, 1), \mathcal{F}, \lambda)$, where $\mathcal{F}$ denotes the σ-algebra generated by all the Lebesgue measurable sets. f is integrable, hence $f \in \mathcal{L}^1$.
Now, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a filtration in $\mathcal{F}$. Again, $\{\frac{k}{2^n} : n \in \mathbb{N}, k = 0, 1, \cdots, 2^n - 1\}$ is dense in $[0, 1)$.
Therefore, $(\mathbb{E}(f|\mathcal{F}_n))_{n\in\mathbb{N}}$ is a uniformly integrable stochastic process.
Let us define,

$$\begin{aligned}
\mathcal{F}_\infty &:= \sigma(\{I_{n,k} : n \in \mathbb{N}, k = 0, 1, \cdots, 2^n - 1\}) \\
&= \sigma(\{\mathcal{F}_n : n \in \mathbb{N}\}).
\end{aligned}$$

Now $\mathbb{E}(f|\mathcal{F}_n)$ is the unique r.v. which is $\mathcal{F}_n$ measurable s.t. $\mathbb{E}(f|\mathcal{F}_n) \to \mathbb{E}(f|\mathcal{F}_\infty) = f$ a.s.. Since f is $\mathcal{F}_\infty$ measurable. Again

$$\mathbb{E}(\mathbb{E}(f|\mathcal{F}_n)|A) = \mathbb{E}(f|A), \quad \forall A \in \mathcal{F}_n.$$

Hence,

$$\begin{aligned}
&\frac{1}{2^n}\mathbb{E}(f|\mathcal{F}_n)(x) = \int_{I_{n,k}} f d\lambda, \quad \forall x \in I_{n,k} \\
&\Rightarrow \mathbb{E}(f|\mathcal{F}_n)(x) = f_n(x).
\end{aligned}$$

Therefore, $\mathbb{E}(f_{n+1}|\mathcal{F}_n) = f_n$ a.s.. This completes the proof.

$\square$

4. Let $\{X_i; i \geq 0\}$ be a sequence of i.i.d random variables and let N be a Poisson r.v. independent of $\{X_i, i \geq 0\}$. Let $Y := X_0 + X_1 + \cdots + X_N$. Show that Y is an infinitely divisible r.v..

Solution: Let $N \sim Poi_\lambda$. Now

$$\begin{aligned}\varphi_Y(t) &= \sum_{k=0}^{\infty} \mathbb{P}[N=k]\mathbb{E}[e^{i\langle t, X_0+\cdots+X_N\rangle}] \\ &= \sum_{k=0}^{\infty} \mathbb{P}[N=k]\varphi_X(t)^k \\ &= \exp(\lambda(\varphi_X(t)-1)).\end{aligned}$$

$\forall m \in \mathbb{N}$, $\varphi_Y(t)$ has an m-th root, $\exp(\frac{\lambda}{m}(\varphi_X(t)-1))$, which is the characteristic function of the r.v. $Y^m := X_0 + \cdots + X_{N^m}$, where $N^m \sim Poi_{\lambda/m}$. Hence, Y is infinitely divisible.

□

5. Let Y be a Binomial random variable with parameters $n \geq 1$ and p. $0 < p < 1$. Is Y infinitely divisible? Justify your answer.

Solution: The Bernoulli distribution is a special case of the Binomial distribution where $n = 1$. So, $X \sim B(1,p)$ is same as $X \sim B(p)$. Conversely, any binomial distribution, $B(n,p)$ is the distribution of the sum of n Bernoulli trials, $B(p)$, each with the same probability p.
The Binomial distribution is not infinitely divisible. To show this, we prove that the Bernoulli distribution is not infinitely divisible.
For $0 < p < 1$, let X be a Bernoulli r.v. with parameter p. We know that the generating function for X is $G(z) = 1 - p + pz$. Now, if X is infinitely divisible, then there must be a r.v. Y with probability generating function

$$G_Y(z) := \sqrt{G(z)} = (1-p+pz)^{1/2}$$

so that $X = Y_1 + Y_2$, where Y_1, Y_2 are i.i.d. with the same distribution as Y. But,

$$G''_Y(z) = \frac{1}{2}\left(-\frac{1}{2}\right)p^2(1-p+pz)^{-3/2};$$

therefore,

$$G''_Y(0) = -\frac{1}{4}p^2(1-p)^{-3/2} < 0,$$

which is not a probability. □

6. Let $0 < \alpha < 2$, and define a measure ν_α on $\mathbb{R}$ by $\nu_\alpha(dx) := \theta_\alpha^{-1}|x|^{-\alpha-1}dx$, where θ_α is a positive constant. Show that ν_α is a canonical measure and hence that there exists an infinitely divisible probability measure μ_α on $\mathbb{R}$ with the canonical triple $(0, 0, \nu_\alpha)$. Show that for a suitable choice of θ_α, $\psi_\alpha(t)$, the logarithm of the characteristic function of μ_α is given as $\psi_\alpha(t) = -|t|^\alpha$.

Solution: i)From the definition of ν_α, it is σ-finite measure. Because, for some $a, \zeta \neq 0$ (either a, ζ both positive or both negative)

$$\int_0^a \theta_\alpha^{-1}|x|^{-\alpha-1}dx = \lim_{\zeta\to 0}\int_\zeta^a \theta_\alpha^{-1}|x|^{-\alpha-1}dx.$$

Since, $[\zeta, a]$ can be written as, $[\zeta, a] = \cup_i A_i$, where $A_i \cap A_j = \emptyset$ for $i \neq j$ s.t. $\int_{A_i} \theta_\alpha^{-1}|x|^{-\alpha-1}dx < \infty$. And at $a = 0$, the integral is zero. Hence, we can conclude that the measure ν_α is σ-finite measure.
ii) $\nu_\alpha(\{0\}) = 0$.
iii) We will show that $\int_{\mathbb{R}}(x^2 \wedge 1)\nu_\alpha(dx) < \infty$.
Now,

$$\int_{\mathbb{R}}(x^2 \wedge 1)\nu_\alpha(dx) = \int_{-1}^{1}(x^2 \wedge 1)\nu_\alpha(dx) + \int_{\mathbb{R}\setminus[-1,1]}(x^2 \wedge 1)\nu_\alpha(dx).$$

Now,

$$\begin{aligned}\int_{-1}^{1}(x^2 \wedge 1)\nu_\alpha(dx) &= 2\int_0^1 x^2\theta_\alpha^{-1}|x|^{-\alpha-1}dx \\ &= 2\int_0^1 \theta_\alpha^{-1}x^{1-\alpha}dx \\ &= \frac{2}{\theta_\alpha}\left[\frac{x^{2-\alpha}}{2-\alpha}\right]_0^1.\end{aligned}$$

Since $\alpha < 2$, therefore $\int_{-1}^{1}(x^2 \wedge 1)\nu_\alpha(dx) < \infty$. Again,

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]}(x^2 \wedge 1)\nu_\alpha(dx) &= \int_{\mathbb{R}\setminus[-1,1]}\nu_\alpha(dx) \\ &= \int_{\mathbb{R}\setminus[-1,1]}\theta_\alpha^{-1}|x|^{-\alpha-1}dx \\ &= 2\frac{1}{\theta_\alpha}\int_1^\infty x^{-\alpha-1}dx \\ &= 2\frac{1}{\theta_\alpha}\left[\frac{x^{-\alpha}}{-\alpha}\right]_1^\infty\end{aligned}$$

Since $\alpha > 0$, therefore $\int_{\mathbb{R}}(x^2 \wedge 1)\nu_\alpha(dx) < \infty$.
(i), (ii), (iii) together imply that ν_α is a canonical measure and hence that there exists an infinitely divisible probability measure μ_α on $\mathbb{R}$ with the canonical triple $(0, 0, \nu_\alpha)$.
If $\psi_\alpha(t)$ be the logarithm of the characteristic function of μ_α with canonical triple $(0, 0, \nu_\alpha)$, then by Lévy-Khinchin formula

$$\psi_\alpha(t) = \int_{-\infty}^{\infty}(e^{itx} - 1 - itx\mathbb{1}_{\{|x|<1\}})\theta_\alpha^{-1}|x|^{-\alpha-1}dx$$

Now

$$e^{itx} = \cos(tx) + i\sin(tx),$$

$$\int_{-\infty}^{\infty} itx\mathbb{1}_{\{|x|<1\}}\theta_\alpha^{-1}|x|^{-\alpha-1}dx = 0,$$

because, $itx\mathbb{1}_{\{|x|<1\}}|x|^{-\alpha-1}$ is odd function and also

$$\int_{-\infty}^{\infty} i\sin(tx)\theta_\alpha^{-1}|x|^{-\alpha-1}dx = 0,$$

because, $\sin(tx)|x|^{-\alpha-1}$ is odd function too. Therefore,

$$\begin{aligned}\psi_\alpha(t) &= \int_{-\infty}^{\infty} (\cos(tx) - 1)\theta_\alpha^{-1}|x|^{-\alpha-1}dx \\ &= -\theta_\alpha^{-1} \int_{-\infty}^{\infty} (1 - \cos(z)) \frac{|z|^{-\alpha-1}}{|t|^{-\alpha-1}} \frac{dz}{t} \\ &= -\theta_\alpha^{-1}|t|^\alpha \int_{-\infty}^{\infty} (1 - \cos(z))|z|^{-\alpha-1}dz.\end{aligned}$$

Now, choose

$$\theta_\alpha := \int_{-\infty}^{\infty} (1 - \cos(z))|z|^{-\alpha-1}dz.$$

Then we can conclude that,

$$\psi_\alpha(t) = -|t|^\alpha.$$

$\square$